Minicourse on Evolutionary Game Theory

Lund University

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Outline

1. The basic stochastic evolutionary model
2. Mean dynamics (and other approximations)
3. Canonical evolutionary dynamics, their families, and their properties
4. Potential games and contractive games
5. Nonconvergence
6. Sampling dynamics for normal form games

Recent survey focusing on applications of EGT, stochastic models: Newton (2018)

1. The basic stochastic evolutionary model

1.1 Games

We consider games played by a single unit-mass population.

$S = \{1, \dots, n\}$ strategies
$X = \{x \in \mathbb{R}^n_+ \colon \sum_{i \in S} x_i = 1\}$ population states/mixed strategies

We sometimes consider the play of a symmetric normal form game $A \in \mathbb{R}^{n \times n}$.

A_{ij} is the payoff for playing $i \in S$ against $j \in S$.

We must specify the nature of the matching process:
complete matching, or occasional random matches?

With complete matching, the interaction can be described as a population game.
(Other sorts of matching will be introduced soon.)

Population games

$S = \{1, \ldots, n\}$	strategies
$X = \{x \in \mathbb{R}^n_+ \colon \sum_{i \in S} x_i = 1\}$	population states/mixed strategies
$F_i \colon X \to \mathbb{R}$	payoffs to strategy i (continuous)
$F \colon X \to \mathbb{R}^n$	payoffs to all strategies

x^* is a Nash equilibrium if

$$x_i^* > 0 \text{ implies that } F_i(x^*) \geq F_j(x^*) \text{ for all } j \in S.$$

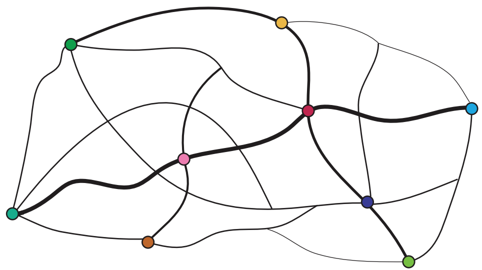
Theorem. *Every population game admits at least one Nash equilibrium.*

Examples of population games

ex. 1. Matching in (symmetric two-player) normal form games

$A \in \mathbb{R}^{n \times n}$	payoff matrix
A_{ij}	payoff for playing $i \in S$ against $j \in S$
$F_i(x) = \sum_j A_{ij} x_j = (Ax)_i$	total payoff for playing i against $x \in X$
$F(x) = Ax$	

ex. 2. Congestion games (Beckmann, McGuire, and Winsten (1956))



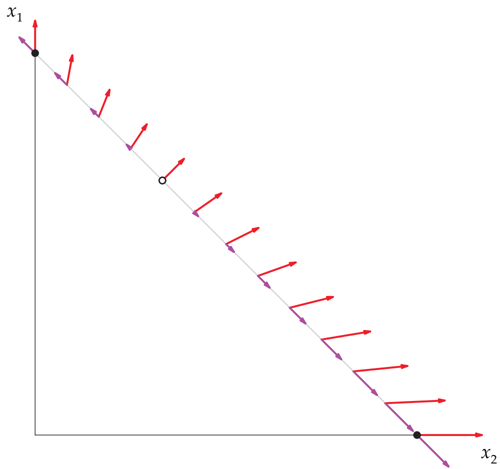
Home and Work are connected by paths $i \in S$ consisting of links $\ell \in \mathcal{L}$.

The payoff to choosing path i is

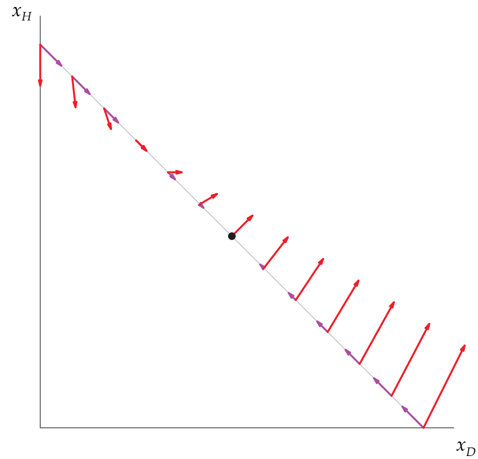
$-$(the delay on path i) = $-$(the sum of the delays on links in path i)

$F_i(x) = -\sum_{\ell \in \mathcal{L}_i} c_\ell(u_\ell(x))$	payoff to path i
x_i	mass of players choosing path i
$u_\ell(x) = \sum_{i:\, \ell \in \mathcal{L}_i} x_i$	utilization of link ℓ
$c_\ell(u_\ell)$	(increasing) cost of delay on link ℓ

The geometry of population games

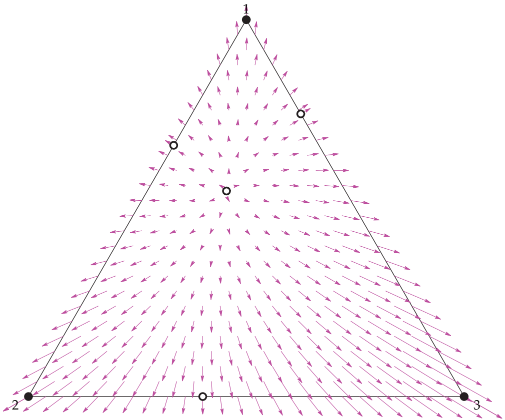


(i) 12 Coordination



(ii) Hawk-Dove

$$F^{C2}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_2 \end{pmatrix}; \qquad F^{HD}(x) = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_H \\ x_D \end{pmatrix} = \begin{pmatrix} 2x_D - x_H \\ x_D \end{pmatrix}.$$



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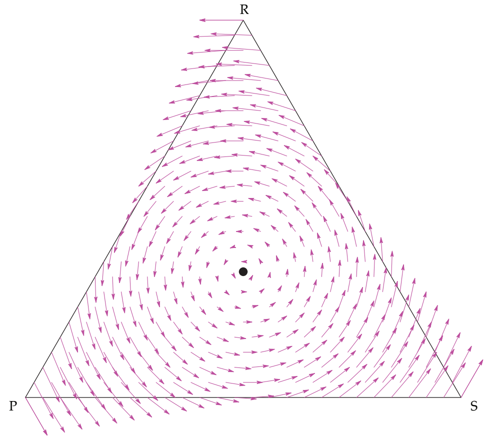


(ii) standard Rock-Paper-Scissors

$$F^{C3}(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 2x_2 \\ 3x_3 \end{pmatrix}; \; F^{RPS}(x) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_R \\ x_P \\ x_S \end{pmatrix} = \begin{pmatrix} x_S - x_P \\ x_R - x_S \\ x_P - x_R \end{pmatrix}.$$

1.2 Revision protocols

Evolutionary dynamics for games reflect two basic assumptions.

Inertia: agents only occasionally consider switching strategies.

This will be captured in the random assignment of revision opportunities.

Myopia: agents base their decisions on the information they have about the current strategic environment.

How agents make decisions is modeled using revision protocols

(Weibull (1995), Björnerstedt and Weibull (1996), Sandholm (2010), IIS (2018)).

Revision protocols

ρ	revision protocol
$\rho^F : X \to \mathbb{R}^{n\times n}_+$	the revision protocol for game F
$\rho^F_{ij}(x)$	conditional switch rate

If $\sum_{j \in S} \rho^F_{ij}(x) = 1$, then $\rho^F_{ij}(x)$ is a conditional switch probability:

If an i player receives a revision opportunity, he switches to j with probability $\rho^F_{ij}(x)$.

3. Decision rules

Let π be a vector of "performances" of the strategies under consideration (two strategies, all strategies, . . .)

Some decision rules mapping π to probabilities of choosing each strategy:

ex. pairwise difference $\quad P_{ij}(\pi_i, \pi_j) = [\pi_j - \pi_i]_+$

ex. maximization $\quad M_j(\pi) = 1 \text{ if } j = \operatorname*{argmax}_{k \in S} \pi_k$

ex. logit (a perturbed max) $\quad L_j(\pi) = \dfrac{\exp(\eta^{-1}\pi_j)}{\sum_k \exp(\eta^{-1}\pi_j)} \quad$ ($\eta > 0$ is the noise level)

1. Imitation vs. direct selection of candidate strategies

ex. pairwise proportional imitation (Helbing (1992), Schlag (1998))

$$\rho_{ij}^{F}(x) = x_j [F_j(x) - F_i(x)]_+$$

ex. pairwise proportional direct selection (Smith (1984))

$$\rho_{ij}^{F}(x) = \frac{1}{n} [F_j(x) - F_i(x)]_+$$

2. Complete vs. limited matching (in normal form game $A \in \mathbb{R}^{n \times n}$)

maximization: $M_j(\pi) = 1$ if $j = \underset{k \in S}{\operatorname{argmax}}\, \pi_k$

ex. optimization with complete matching (Gilboa-Matsui (1991), Hofbauer (1995))

$$\rho_{ij}^A(x) = M_j(Ax).$$

ex. sample best response (Sandholm (2001), KDV (2002), OST (2015))

$$\rho_{ij}^A(x) = \sum_{k \in S} x_k M_j(A^k). \qquad (A^k = k\text{th column of } A)$$

ex. best experienced payoff (Sethi (2000), IIS (2018))

$$\rho_{ij}^A(x) = x_j \sum_{k \in S} \sum_{\ell \in S} x_k x_\ell 1[A_{ik} < A_{j\ell}].$$

A general formulation of revision protocols (for normal form games) (IIS 2018)

$$\rho_{ij}^{A}(x) = \sum_{(s,\pi)} p_i(s \,|\, x)\, q(\pi \,|\, s, x, A)\, \sigma_{ij}(s, \pi).$$

1. $p_i(s \,|\, x)$ describes selection of candidate strategies $s = (s_1, \ldots, s_c)$.

2. $q(\pi \,|\, s, x, A)$ describes how payoffs $\pi = (\pi_1, \ldots, \pi_c)$ are associated with candidate strategies through matching/sampling.

3. $\sigma_{ij}(s, \pi)$ describes the decision rule.

I: Imitative protocols

decision method		matching: complete matching	matching: limited matching
	pairwise difference	replicator [a]	
	linear-*	replicator [b]	
	best	imitate the best [c]	imitate the best realization [d]
	logit	imitative logit [e]	—
	positive proportional	Maynard Smith replicator [f]	—

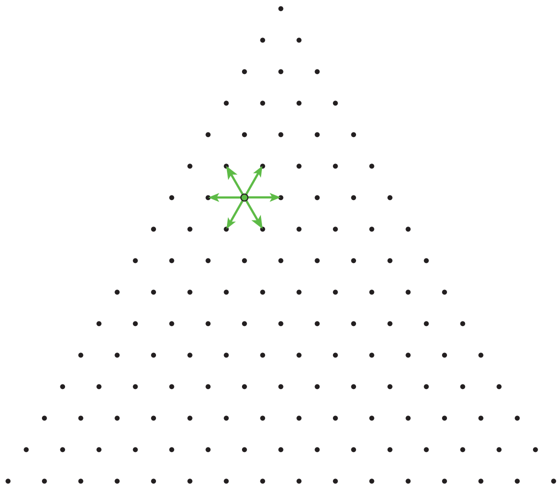
II: Direct protocols

decision method		matching/sampling: complete matching	limited matching: single sample	limited matching: multiple samples
	pairwise difference	Smith [g]	—	—
	linear-*	—	—	—
	best	best response [h]	sample best response [i]	best experienced payoff [j]
	logit	logit [k]	sample logit [l]	—
	positive proportional	—	—	—

[a] Taylor and Jonker (1978); Helbing (1992); Schlag (1998); Hofbauer (1995a).
[b] Taylor and Jonker (1978); Weibull (1995); Björnerstedt and Weibull (1996).
[c] Hofbauer (1995a); Vega-Redondo (1997).
[d] Izquierdo and Izquierdo (2013).
[e] Weibull (1995).
[f] Maynard Smith (1982); Binmore et al. (1995); Fudenberg et al. (2006); Fudenberg and Imhof (2008); Sandholm (2010b).
[g] Smith (1984); Sandholm (2010a).
[h] Gilboa and Matsui (1991); Hofbauer (1995b); Kandori and Rob (1995).
[i] Sandholm (2001); Kosfeld et al. (2002); Oyama et al. (2015).
[j] Sethi (2000); Sandholm et al. (2017).
[k] Blume (1997); Fudenberg and Levine (1998); Hofbauer and Sandholm (2007).
[l] Kreindler and Young (2013).

1.3 The stochastic evolutionary process

A population game F, a revision protocol ρ, and a finite population size N together define a stochastic evolutionary process $\{X_k^N\}_{k\geq 0}$ on the discrete grid $\mathcal{X}^N = X \cap \frac{1}{N}\mathbb{Z}^n$.



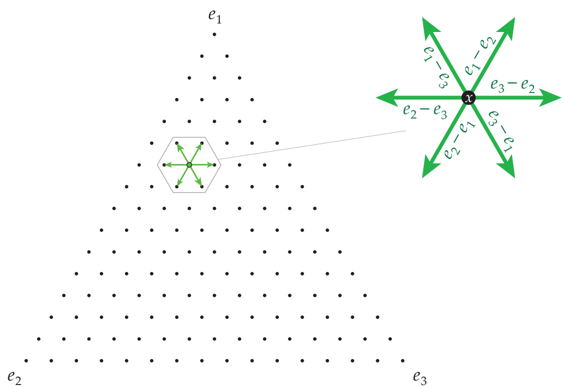
Each period takes $\frac{1}{N}$ units of clock time.

During a period, one agent is randomly assigned a revision opportunity

If he is playing $i \in S$, he switches to j with probability $\rho_{ij}^F(x)$.

The transition probabilities for the process $\{X_k^N\}_{t\geq 0}$ are given by

$$\mathbb{P}\left(X_{k+1}^N = y \,\middle|\, X_k^N = x\right) = \begin{cases} x_i\, \rho_{ij}^F(x) & \text{if } y = x + \frac{1}{N}(e_j - e_i),\, j \neq i \\ \sum_{i=1}^n x_i\, \rho_{ii}^F(x) & \text{if } y = x. \end{cases}$$



2. Mean dynamics (and other approximations)

For tractability, we consider the behavior of $\{X_t^N\}$ over $[0,T]$ as N grows large.

Over the next dt time units ($= N\,dt$ periods), starting from state x:

the number of revision opportunities to arrive:	$N\,dt$
the expected number received by current i players:	$Nx_i\,dt$
the expected number of these leading to switches to j:	$Nx_i\,\rho_{ij}^F(x)\,dt$

$\therefore$ the expected change in the *proportion* of agents using strategy i is

$$\left(\sum_{j\in S} x_j \rho_{ji}^F(x) - x_i \sum_{j\in S} \rho_{ij}^F(x)\right) dt.$$

The mean dynamic induced by F and ρ is thus

$$\text{(M)} \qquad \dot{x} = V(x), \quad \text{where } V_i(x) = \sum_{j\in S} x_j \rho_{ji}^F(x) - x_i \sum_{j\in S} \rho_{ij}^F(x).$$

2.1 Examples

Let $\bar{F}(x) = \sum_{k \in S} x_k F_k(x)$ denote the average payoffs in the population in F at x.

ex. pairwise proportional imitation

1. in population games (e.g., complete matching in A) (Helbing (1992))

$$\rho_{ij}^F(x) = x_j [F_i(x) - F_j(x)]_+ \qquad \text{(e.g., } F_i(x) = (Ax)_i\text{)}$$

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$$\begin{aligned}\dot{x}_i &= \sum_{j\in S} x_j \rho_{ji}^F(x) - x_i \sum_{j\in S} \rho_{ij}^F(x) \\ &= \sum_{j\in S} x_j x_i [F_i(x) - F_j(x)]_+ - x_i \sum_{j\in S} x_j [F_j(x) - F_i(x)]_+\end{aligned}$$

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$$\begin{aligned}
\dot{x}_i &= \sum_{j\in S} x_j \rho_{ji}^F(x) - x_i \sum_{j\in S} \rho_{ij}^F(x) \\
&= \sum_{j\in S} x_j x_i [F_i(x) - F_j(x)]_+ - x_i \sum_{j\in S} x_j [F_j(x) - F_i(x)]_+ \\
&= x_i \sum_{j\in S} x_j (F_i(x) - F_j(x)) \\
&= x_i \left(F_i(x) - \bar{F}(x)\right).
\end{aligned}$$

This is the replicator dynamic of Taylor and Jonker (1978).

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&= x_i \left(F_i(x) - \bar{F}(x)\right).
\end{aligned}$$

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Under complete matching in a normal form game ($F(x) = Ax$), this becomes

$$\dot{x}_i = x_i((Ax)_i - x'Ax).$$

2. under single matches in a normal form game (Schlag (1998))

$$\rho_{ij}^{A}(x) = x_j \sum_{k\in S} \sum_{\ell\in S} x_k x_\ell [A_{j\ell} - A_{ik}]_+$$

2. under single matches in a normal form game (Schlag (1998))

$$\rho_{ij}^{A}(x) = x_j \sum_{k \in S} \sum_{\ell \in S} x_k x_\ell [A_{j\ell} - A_{ik}]_+$$

$$\begin{aligned}
\dot{x}_i &= \sum_{j \in S} \left(x_j \rho_{ji}^{A}(x) - x_i \rho_{ij}^{A}(x) \right) \\
&= \sum_{j \in S} \left(x_j \left(x_i \sum_{k \in S} \sum_{\ell \in S} x_k x_\ell [A_{ik} - A_{j\ell}]_+ \right) - x_i \left(x_j \sum_{k \in S} \sum_{\ell \in S} x_k x_\ell [A_{j\ell} - A_{ik}]_+ \right) \right) \\
&= x_i \sum_{j \in S} x_j \sum_{k \in S} \sum_{\ell \in S} x_k x_\ell (A_{ik} - A_{j\ell}) \\
&= x_i \sum_{j \in S} x_j ((Ax)_i - (Ax)_j) \\
&= x_i ((Ax)_i - x' Ax).
\end{aligned}$$

The replicator dynamic again!

2.2 Finite horizon deterministic approximation

What does the mean dynamic tell us about the original stochastic process $\{X_t^N\}$?

Theorem (Benaïm-Weibull (2003))**.**

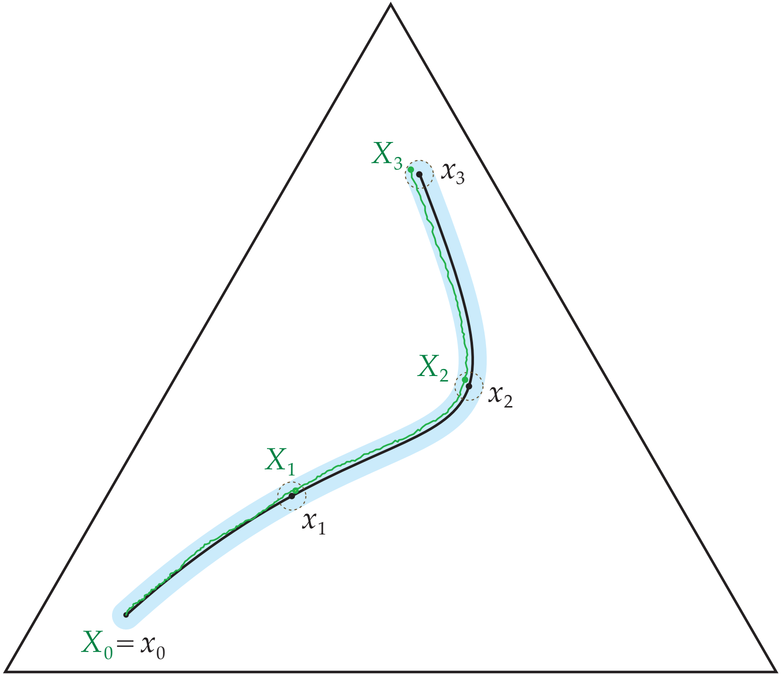
Suppose that the mean dynamic V is Lipschitz continuous.

Let the initial conditions $X_0^N = x_0^N$ converge to state $x_0 \in X$.

Let $\{x_t\}_{t \geq 0}$ be the solution to the mean dynamic $\dot{x} = V(x)$ starting from x_0.

Then for all $T < \infty$ and $\varepsilon > 0$,

$$\lim_{N \to \infty} \mathbb{P}\left(\sup_{t \in [0,T]} \left| X_t^N - x_t \right| < \varepsilon \right) = 1.$$



finite horizon deterministic approximation

2.3 Diffusion approximation near rest points

Let $x^* \in X$ be a rest point of (M): $V(x^*) = 0$.

Then in the large N limit, there are no changes in aggregate behavior at x^*.

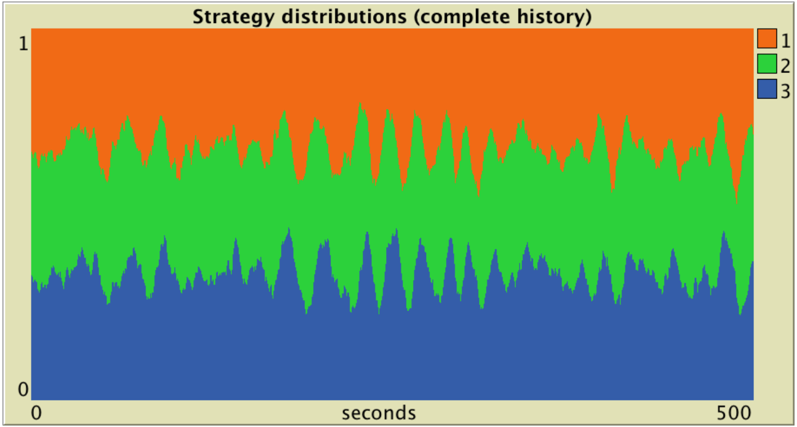
But agents may still be switching between strategies!

If N is not too large, one can obtain a "central limit theorem" approximation of aggregate behavior near locally stable rest points of (M).

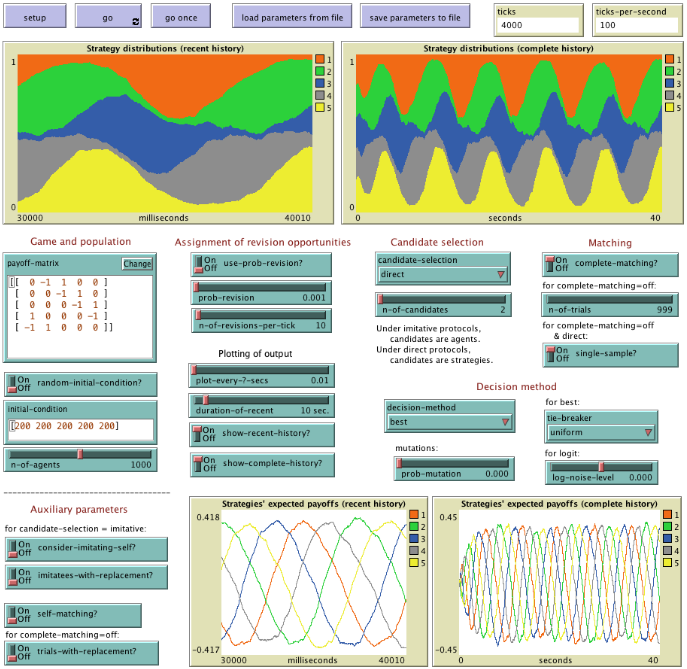
Variation around the equilibrium is of order $1/\sqrt{N}$.

The variation need not be negligible, even in fairly large populations.

(See Sandholm (2003) and IIS (2018).)



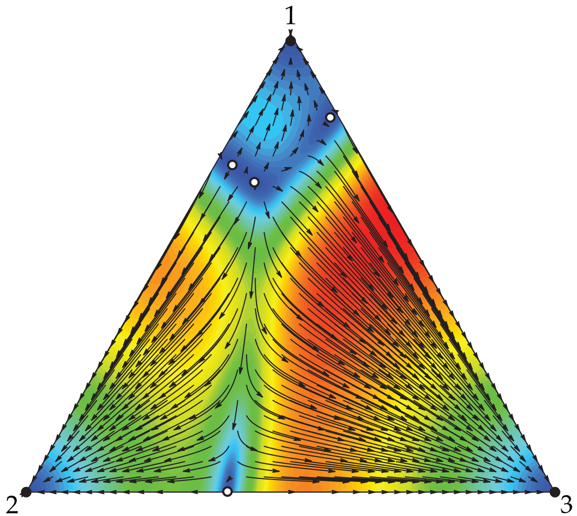
imitative linear-dissatisfaction in good Rock-Paper-Scissors; complete matching; 1000 agents



Simulations using *ABED* (IIS (2018))

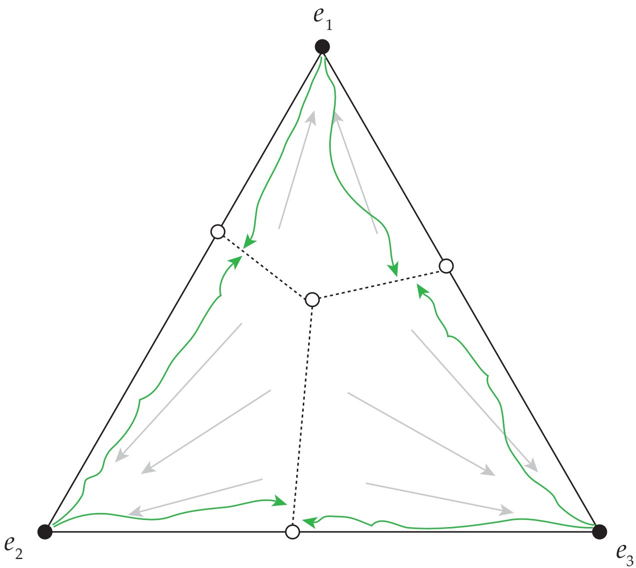
2.4 Large deviations and equilibrium selection

Mean field approximations describe typical behavior of $\{X_t^N\}$.

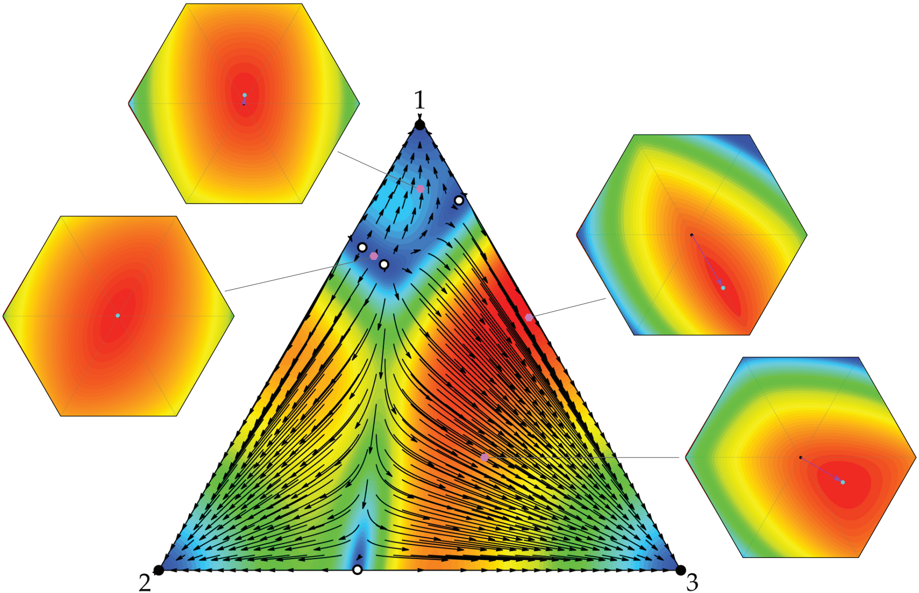


$$F(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \quad \rho_{ij}^F(x) = \frac{\exp(\eta^{-1} F_j(x))}{\sum_{k \in S} \exp(\eta^{-1} F_k(x))}, \ \eta = .2.$$

Large deviations theory describes excursions between attractors.



Cramér transforms $L(x,z)$ measure "difficulty" of motion from x in directions z differing from the expected motion $V(x)$.



See Sandholm and Staudigl (2018).

Large deviations are of direct interest as a description of equilibrium breakdown.

They also provide the starting point for stochastic stability analysis,
which describes the time-averaged behavior of $\{X_t^N\}$
over very long time spans.
(Foster-Young (1990), KMR (1993), Young (1993, 1998)).

3. Canonical evolutionary dynamics, their families, and their properties

Returning to deterministic dynamics. . .

The traditional deterministic dynamics concern population games, including complete matching in normal form games.

3.1 Definition of evolutionary dynamics via revision protocols

Let $\mathcal{F}$ be a set of population games $F\colon X \to \mathbb{R}^n$.

Let $\mathcal{D}$ be the set of nice ODEs $\dot{x} = V(x)$ on X.

A map that assigns each game $F \in \mathcal{F}$ a differential equation in $\mathcal{D}$ is called a deterministic evolutionary dynamic.

Every nice revision protocol ρ implicitly defines a deterministic evolutionary dynamic via

$$\text{(M)} \qquad \dot{x} = V^F(x), \quad \text{where } V_i^F(x) = \sum_{j \in S} x_j \rho_{ji}^F(x) - x_i \sum_{j \in S} \rho_{ij}^F(x).$$

3.2 Four canonical evolutionary dynamics

revision protocol	mean dynamic	name
$\rho_{ij} = x_j[\pi_j - \pi_i]_+$	$\dot{x}_i = x_i(F_i(x) - \bar{F}(x))$	replicator
$\rho_{ij} = [\pi_j - \pi_i]_+$	$\dot{x}_i = \sum_{j\in S} x_j[F_i(x) - F_j(x)]_+ \quad - x_i \sum_{j\in S} [F_j(x) - F_i(x)]_+$	Smith
$\rho_{ij} = M_j(\pi)$	$\dot{x} \in M(F(x)) - x$	best response
$\rho_{ij} = \dfrac{\exp(\eta^{-1}\pi_j)}{\sum_{k\in S}\exp(\eta^{-1}\pi_k)}$	$\dot{x}_i = \dfrac{\exp(\eta^{-1}F_i(x))}{\sum_{k\in S}\exp(\eta^{-1}F_k(x))} - x_i$	logit(η)

replicator: Taylor-Jonker (1978)

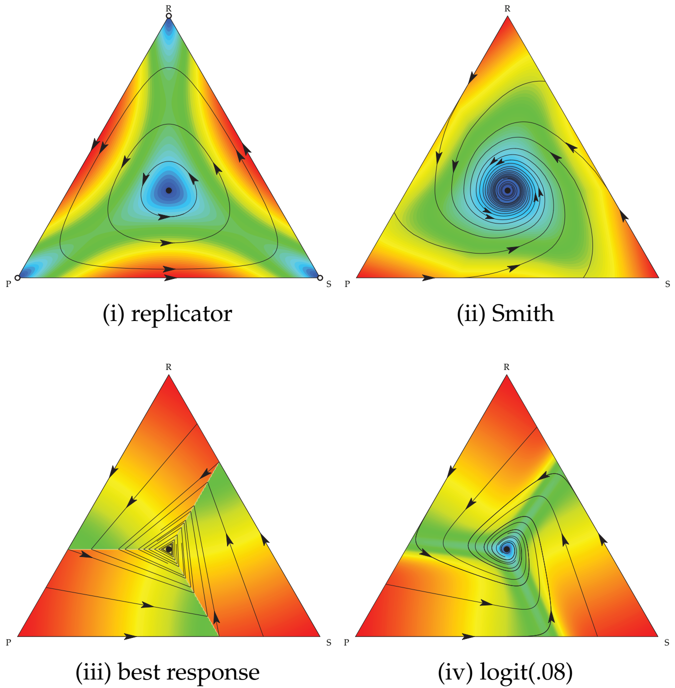
Smith: Smith (1984)

best response: Gilboa-Matsui (1991), Hofbauer (1995)

logit: Fudenberg-Levine (1998)

Example ((Standard) Rock-Paper-Scissors).

$$A = \begin{pmatrix} 0 & -\ell & w \\ w & 0 & -\ell \\ -\ell & w & 0 \end{pmatrix}, \quad \text{where } w = \ell > 0.$$



(i) replicator (ii) Smith

(iii) best response (iv) logit(.08)

The canonical dynamics in standard RPS. Red is fastest, blue is slowest.

New, friendly software for drawing diagrams: *EvoDyn-3s* (IIS 2018)

Older, more elaborate software: *Dynamo* (SDF 2014)

3.3 Incentives and aggregate behavior

We introduce conditions that relate the evolution of aggregate behavior under the dynamics to the incentives in the underlying game.

These conditions are key tools for proving convergence results.

(PC) *Positive correlation* $V(x) \neq \mathbf{0} \;\Rightarrow\; V(x)'F(x) > 0.$

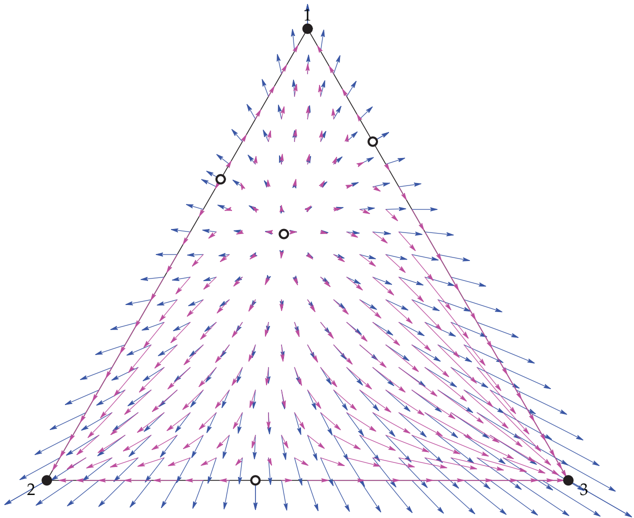
(NS) *Nash stationarity* $V(x) = \mathbf{0} \;\Leftrightarrow\; x \in NE(F).$

Game-theoretic interpretation of (PC):

Requires a positive correlation between growth rates and payoffs under the uniform probability distribution on strategies.

Geometric interpretation of (PC): $V(x) \neq \mathbf{0} \;\Rightarrow\; V(x)'F(x) > 0$

If the growth rate vector $V(x)$ is nonzero,
it forms an acute angle with the payoff vector $F(x)$.



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Interpretation of (NS): no aggregate motion if and only if Nash equilibrium.

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The ($\Leftarrow$) direction follows from (PC):

Proposition. *If V satisfies* (PC)*, then $x \in NE(F)$ implies that $V(x) = \mathbf{0}$.*

Why? Consider $x \in \text{int}(X)$.

Then $x \in NE(F)$ means that $F_i(x) = \pi^*$ for all $i \in S$.

If $V(x)$ is a feasible direction of motion, then $\sum_{i\in S} V_i(x) = 0$.

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Then $x \in NE(F)$ means that $F_i(x) = \pi^*$ for all $i \in S$.

If $V(x)$ is a feasible direction of motion, then $\sum_{i \in S} V_i(x) = 0$.

$$V(x)'F(x) = \sum_{i \in S} V_i(x) F_i(x) = \pi^* \sum_{i \in S} V_i(x) = 0.$$

Thus (PC) $\Rightarrow$ $V(x) = \mathbf{0}$.

(PC)	*Positive correlation*	$V(x) \neq \mathbf{0} \;\Rightarrow\; V(x)'F(x) > 0.$
(NS)	*Nash stationarity*	$V(x) = \mathbf{0} \;\Leftrightarrow\; x \in NE(F).$

Interpretation of (NS): no aggregate motion if and only if Nash equilibrium.

The ($\Leftarrow$) direction follows from (PC):

Proposition. *If V satisfies* (PC)*, then $x \in NE(F)$ implies that $V(x) = \mathbf{0}$.*

Why? Consider $x \in \mathrm{int}(X)$.

Then $x \in NE(F)$ means that $F_i(x) = \pi^*$ for all $i \in S$.

If $V(x)$ is a feasible direction of motion, then $\sum_{i \in S} V_i(x) = 0$.

$$V(x)'F(x) = \sum_{i \in S} V_i(x) F_i(x) = \pi^* \sum_{i \in S} V_i(x) = 0.$$

Thus (PC) $\;\Rightarrow\;$ $V(x) = \mathbf{0}$.

For $x \in \mathrm{bd}(X)$, NE is less restrictive, but feasibility is more restrictive.

3.4 The replicator dynamic and other imitative dynamics

The replicator dynamic:

(R) $\dot{x}_i = x_i(F_i(x) - \bar{F}(x))$

A more general class of imitative dynamics:

(I) $\dot{x}_i = x_i G_i(x),$ where

(MP) $G_i(x) \geq G_j(x)$ if and only if $F_i(x) \geq F_j(x)$.

If strategy $i \in S$ is in use, then $G_i(x) = \dot{x}_i(x)/x_i$ is the percentage growth rate of the number of agents using i.

Condition (MP) is called monotonicity of percentage growth rates.

It holds for dynamics generated by many imitative protocols:

formula	restriction	interpretation
$\rho_{ij}(\pi, x) = x_j\, \phi(\pi_j - \pi_i)$	$\text{sgn}(\phi(d)) = \text{sgn}([d]_+)$	imitation via pairwise comparisons
$\rho_{ij}(\pi, x) = a(\pi_i)\, x_j$	a decreasing	imitation driven by dissatisfaction
$\rho_{ij}(\pi, x) = x_j\, c(\pi_j)$	c increasing	imitation of success
$\rho_{ij}(\pi, x) = \dfrac{x_j\, w(\pi_j)}{\sum_{k \in S} x_k\, w(\pi_k)}$	w increasing	imitation of success with repeated sampling

(See Weibull (1995), Hofbauer (1995), Sandholm (2010).)

Theorem.

Imitative dynamics (I) *respecting* (MP) *satisfy positive correlation* (PC).

Idea of proof: Under (MP), all strategies with positive growth rates have higher payoffs than all strategies with negative growth rates.

Thus by the proposition, all Nash equilibria are rest points of (I) under (MP).

But there are additional rest points. For example, every pure state is a rest point.

The rest points of (I) under (MP) are the restricted equilibria of F by

$$RE(F) = \{x \in X\colon x_i > 0 \Rightarrow F_i(x) = \max_{j \in S: x_j > 0} F_j(x)\}.$$

But:

Theorem. *Non-Nash rest points of imitative dynamics cannot be locally stable, and cannot be limits of interior solution trajectories.*

(See Bomze (1986) and Nachbar (1990).)

3.5 The Smith dynamic and other direct pairwise comparison dynamics

$$\dot{x}_i = \sum_{j \in S} x_j [F_i(x) - F_j(x)]_+ - x_i \sum_{j \in S} [F_j(x) - F_i(x)]_+.$$

Theorem. *The Smith dynamic satisfies* (PC) *and* (NS).

This result extends to dynamics based on any revision protocol satisfying sign preservation:

$$\operatorname{sgn}(\rho_{ij}(\pi, x)) = \operatorname{sgn}([\pi_j - \pi_i]_+) \quad \text{for all } i, j \in S.$$

It also extends to hybrids of direct and imitative dynamics.

(See Sandholm (2010).)

3.6 The best response dynamic

$$\text{(B)} \qquad \dot{x} \in M(F(x)) - x, \qquad \text{where } M(\pi) = \operatorname*{argmax}_{y \in X} y'\pi.$$

M is set-valued and discontinuous, so (B) is a differential inclusion.

The standard results on existence and uniqueness of solutions do not apply. But:

Theorem. *From every initial condition $x_0 \in X$,* (B) *admits a Carathéodory solution: a Lipschitz continuous trajectory $\{x_t\}_{t \geq 0}$ that satisfies $\dot{x}_t \in V(x_t)$ for almost all $t \geq 0$.*

Solutions from x_0 need be unique.

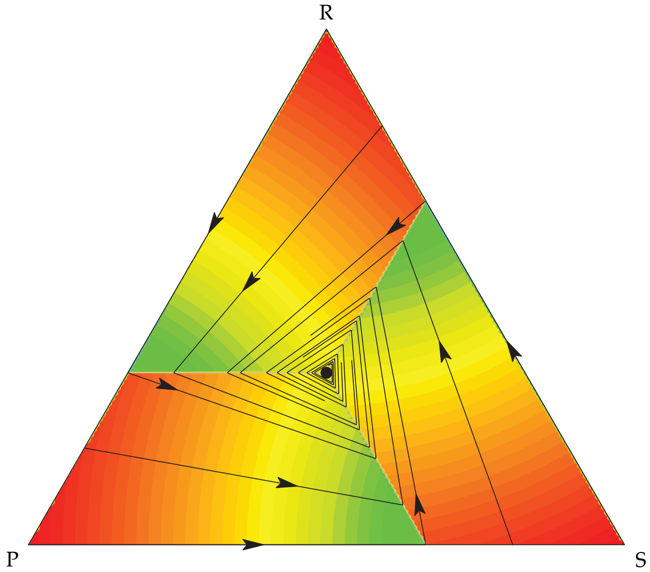
The best response dynamic satisfies versions of (PC) and (NS)
suitable for differential inclusions.

Solutions to (B) are generally not unique.

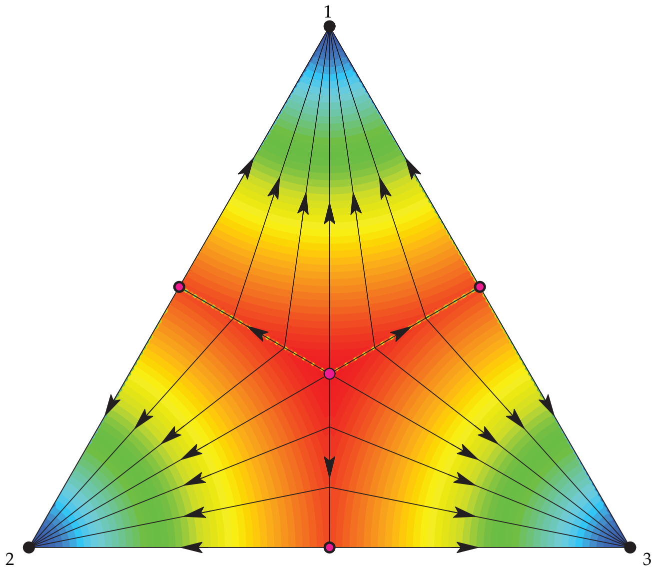
But in regions where the best response is unique, solutions take a simple form:

$$M(F(x)) = \{e_i\} \quad \Rightarrow \quad \dot{x} = e_i - x \quad \Rightarrow \quad x_t = (1 - \mathrm{e}^{-t})\, e_i + \mathrm{e}^{-t}\, x_0.$$

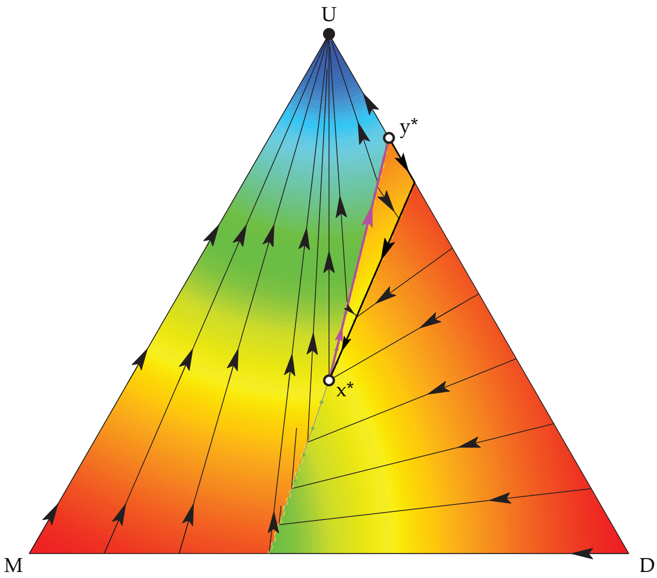
Typically, solutions move from the current state directly toward the vertex corresponding to the current best response.



standard RPS: $A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$

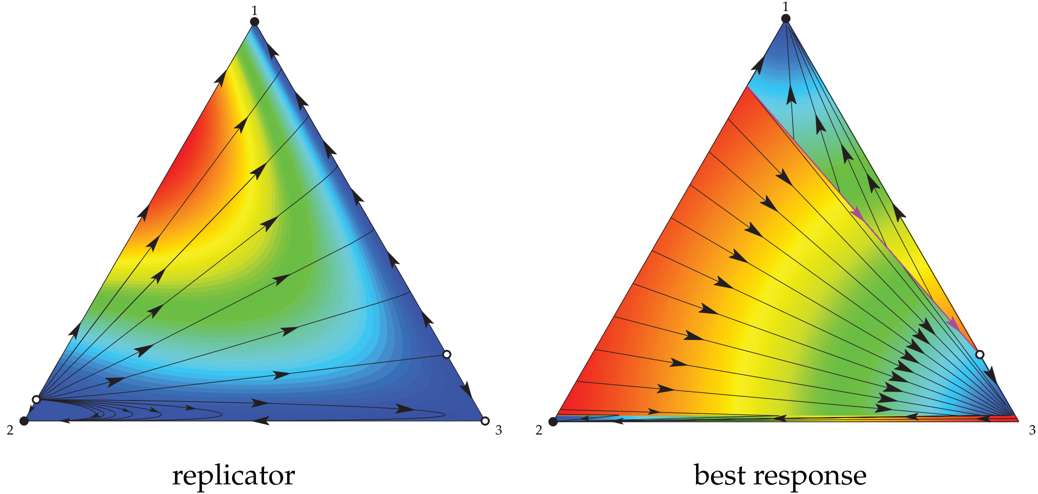


pure coordination: $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$



Zeeman's (1980) game: $A = \begin{pmatrix} 0 & 6 & -4 \\ -3 & 0 & 5 \\ -1 & 3 & 0 \end{pmatrix}$

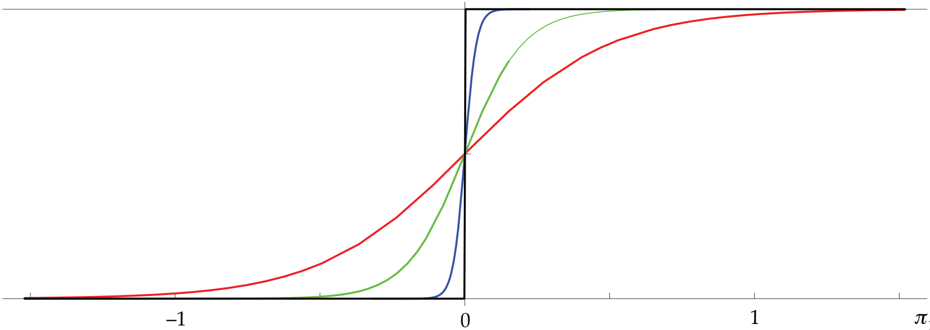
Different revision protocols can lead to different predictions from most initial states.



the Golman-Page (2009) game: $A = \begin{pmatrix} 1 & -k & -\frac{1}{k} \\ 2-k^3 & 2 & 2 \\ 0 & 0 & 0 \end{pmatrix}.$

3.7 The logit dynamic and perturbed best response dynamics

A perturbed maximizer function $\tilde{M}\colon \mathbb{R}^n \to \mathrm{int}(X)$ is a smooth approximation of M.



A general formulation for $\tilde{M}$ (Fudenberg-Levine (1998)):

$$\tilde{M}(\pi) = \operatorname*{argmax}_{y \in \mathrm{int}(X)} (y'\pi - v(y)), \quad v \text{ strongly convex, steep near } \mathrm{bd}(X).$$

The corresponding dynamics are called perturbed best response dynamics:

$$\dot{x} = \tilde{M}(F(x)) - x.$$

These dynamics satisfy appropriately perturbed versions of (PC) and (NS).

(See Hofbauer and Sandholm (2002, 2007).)

Logit choice and the logit dynamic

The logit choice function with noise level η:

$$L_i(\pi) = \frac{\exp(\eta^{-1}\pi_i)}{\sum_{j\in S}\exp(\eta^{-1}\pi_j)}.$$

Derivation:

$$L(\pi) = \operatorname*{argmax}_{y\in \operatorname{int}(X)} (y'\pi - v(y)), \quad v = -\text{entropy} \quad \Big(v(y) = \eta \sum_{j\in S} y_j \log y_j\Big).$$

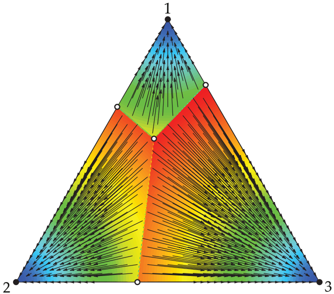
The logit dynamic with noise level η:

$$\text{(L)} \qquad \dot{x}_i = \frac{\exp(\eta^{-1}F_i(x))}{\sum_{j\in S}\exp(\eta^{-1}F_j(x))} - x_i\ .$$

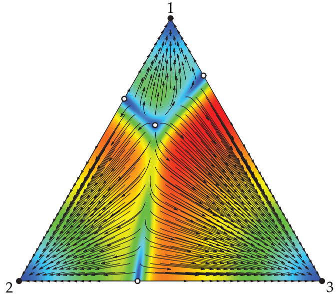
Rest points are called logit equilibria (or quantal response equilibria).

Example: Logit dynamics in 123 Coordination for various noise levels η.

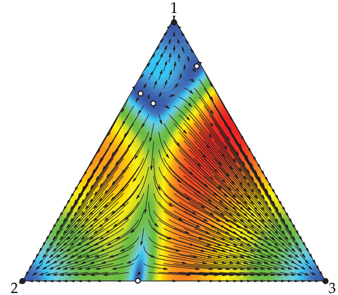
$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$



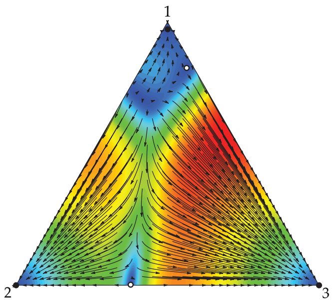
(i) η = .001



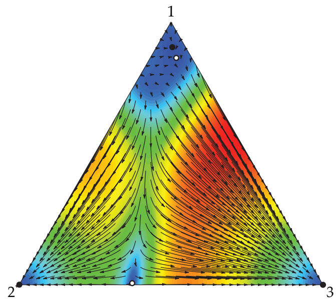
(ii) η = .1



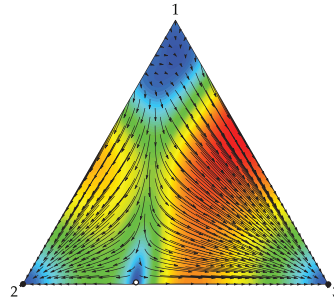
(iii) η = .2



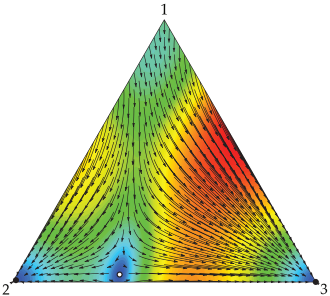
(iv) η = .22



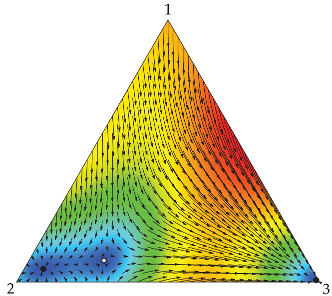
(v) η = .27



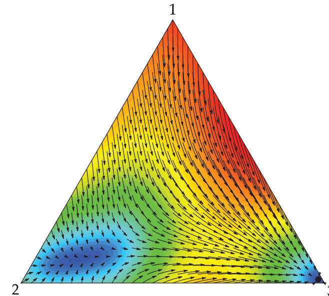
(vi) η = .28



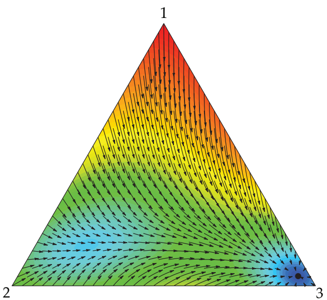
(vii) $\eta = .4$



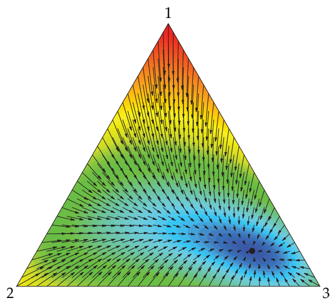
(viii) $\eta = .6$



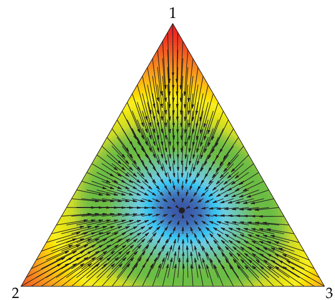
(ix) $\eta = .68$



(x) $\eta = .85$



(xi) $\eta = 1.2$



(xii) $\eta = 3$

Some appealing features of processes based on logit L/perturbed maximization $\tilde{M}$

- Approximates exact maximization to any desired degree.
- $\tilde{M}$ is smooth, allowing analysis of mean dynamics via linearization
- $\tilde{M}$ defined in terms of convex penalty v, allowing use of tools from convex analysis
- Surprising connections between learning processes based on L (fictitious play, reinforcement learning) and imitative dynamics (Hopkins (2002), HSV (2009), Mertikopoulos and Sandholm (2016, 2018))

Returning to the original stochastic process $\{X_t^N\}$:

Perturbed maximization provides the natural environment
for large deviations and stochastic stability analysis.

(See Blume (1997, 2003), Alós-Ferrer and Netzer (2010), Sandholm (2010), Sandholm and Staudigl (2016, 2018))

example	family	continuity	(PC)	(NS)
replicator	imitation	yes	yes	no
Smith	direct pairwise comparison	yes	yes	yes
best response	optimization	no	yes	yes*
logit	perturbed optimization	yes*	approx.	approx.

Canonical dynamics, their families, and their properties.

4. Potential games and contractive games

These are the classes of games with the strongest convergence properties under the canonical dynamics.

Other classes allowing global convergence results:

Supermodular games (for best response and perturbed best response dynamics).

Dominance solvable games (for imitative and best response dynamics).

4.1 Potential games

In potential games, all information about incentives can be captured by a scalar-valued function on the set X of population states.

Dynamics satisfying positive correlation (PC) ascend this function and converge to its maximizers, which are Nash equilibria of the game.

(Hofbauer-Sigmund (1988), Monderer-Shapley (1996), Sandholm (2001)) ex. 1. Common interest games

$A \in R^{n\times n}$ is a common interest game if $A_{ij} = A_{ji}$ for all $i, j \in S$ (i.e., if A is a symmetric matrix).

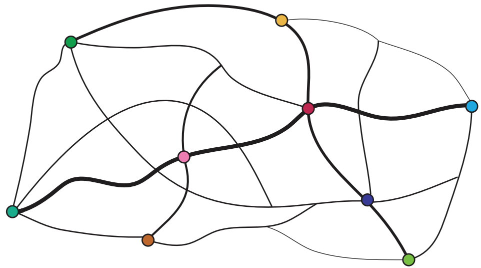
Various economic applications.

Used in biology to model genetic competition.

Theorem (Fundamental Theorem of Natural Selection). *In a common interest game A, solution trajectories of the replicator dynamic increase average fitness $\bar{F}(x) = x'Ax$.*

Versions of this result date back to R. A. Fisher (1930).

ex. 2. Congestion games (Beckmann, McGuire, and Winsten (1956))



Home and Work are connected by paths $i \in S$ consisting of links $\ell \in \mathcal{L}$.

The payoff to choosing path i is

−(the delay on path i) = −(the sum of the delays on links in path i)

$$F_i(x) = -\sum_{\ell \in \mathcal{L}_i} c_\ell(u_\ell(x)) \qquad \text{payoff to path } i$$

BMW observe that simulated adjustment processes converge to Nash equilibrium.

But Nash equilibria of congestion games are typically not efficient.

4.1.1 Definition and characterization

We call F a potential game if there exists a C^1 function $f\colon \mathbb{R}^n_+ \to \mathbb{R}$, called a potential function, satisfying

$$\nabla f(x) = F(x) \text{ for all } x \in X, \text{ or equivalently}$$
$$\frac{\partial f}{\partial x_i}(x) = F_i(x) \text{ for all } i \in S \text{ and } x \in X.$$

If F is smooth on $\mathbb{R}^n_+$, it is a potential game if and only if it satisfies externality symmetry:

$$DF(x) \text{ is symmetric for all } x \in \mathbb{R}^n_+, \text{ or equivalently}$$
$$\frac{\partial F_i}{\partial x_j}(x) = \frac{\partial F_j}{\partial x_i}(x) \text{ for all } i, j \in S \text{ and } x \in \mathbb{R}^n_+.$$

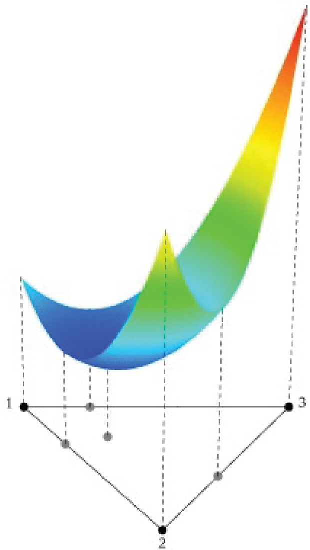
Examples of potential games: $F(x) = \nabla f(x)$; $DF(x)$ symmetric

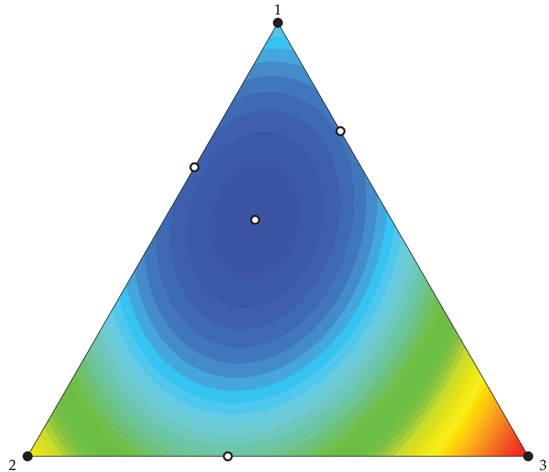
ex. 1. common interest games

$$F(x) = Ax, \; A \text{ symmetric} \quad \Rightarrow \quad f(x) = \frac{1}{2}x'Ax$$

$$DF(x) = A$$

Example: A common interest game and its potential function





$$F(x) = \begin{pmatrix} F_1(x) \\ F_2(x) \\ F_3(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}; \quad f(x) = \tfrac{1}{2}\left((x_1)^2 + 2(x_2)^2 + 3(x_3)^2\right).$$

Potential games: $F(x) = \nabla f(x)$; $DF(x)$ symmetric

ex. 2. congestion games

$$F_i(x) = -\sum_{\ell \in \mathcal{L}_i} c_\ell(u_\ell(x)) \quad \Rightarrow \quad f(x) = -\sum_{\ell \in \mathcal{L}} \int_0^{u_\ell(x)} c_\ell(z)\,\mathrm{d}z$$

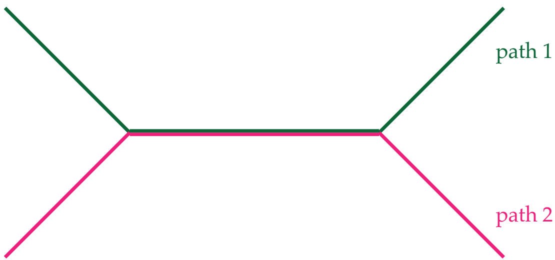
$$\neq -\sum_{\ell \in \mathcal{L}} u_\ell(x) c_\ell(u_\ell(x)) = \bar{F}(x).$$

Potential games: $F(x) = \nabla f(x)$; $DF(x)$ symmetric

ex. 2. congestion games

$$F_i(x) = -\sum_{\ell \in \mathcal{L}_i} c_\ell(u_\ell(x)) \quad \Rightarrow \quad f(x) = -\sum_{\ell \in \mathcal{L}} \int_0^{u_\ell(x)} c_\ell(z)\,\mathrm{d}z$$
$$\neq -\sum_{\ell \in \mathcal{L}} u_\ell(x) c_\ell(u_\ell(x)) = \bar{F}(x).$$

What about externality symmetry: $\dfrac{\partial F_i}{\partial x_j}(x) = \dfrac{\partial F_j}{\partial x_i}(x)$?



Intuition about potential functions:

Suppose that some members of the population switch from strategy i to strategy j, so that the state moves in direction $z = e_j - e_i$.

If these switches improve the payoffs of those who switch, then

$$\frac{\partial f}{\partial z}(x) = \nabla f(x)'z = F(x)'z = F_j(x) - F_i(x) > 0.$$

Thus profitable strategy revisions increase potential.

4.1.2 Global convergence in potential games

Lemma. *Let F be a potential game with potential function f, and suppose the dynamic V satisfies* (PC). *Then along any solution $\{x_t\}$, we have $\frac{\mathrm{d}}{\mathrm{d}t} f(x_t) > 0$ whenever $\dot{x}_t \neq \mathbf{0}$.*

That is, f is a strict Lyapunov function for V.

Proof: $\dfrac{\mathrm{d}}{\mathrm{d}t} f(x_t) = \nabla f(x_t)' \dot{x}_t = F(x_t)' V(x_t) \geq 0$, with equality only if $V(x_t) = \mathbf{0}$.

Global convergence in potential games

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The lemma and basic results on Lyapunov functions yield

Theorem. *Let F be a potential game, and let $\dot{x} = V(x)$ be a satisfy* (PC). *Then all ω-limit points of V are rest points of V. In particular,*

(*i*) *If V is an imitative dynamic, then $\Omega(V^F) = RE(F)$.*
(*ii*) *If V^F is a direct pairwise comparison dynamic, then $\Omega(V^F) = NE(F)$.*

Analogous results hold for exact and perturbed best response dynamics.

4.1.3 More results about potential games

Nash equilibria = states satisfying KKT first-order conditions for maximizing f

f strictly concave $\Rightarrow$ equilibrium is unique and globally stable

f is homogeneous $\Rightarrow$ locally stable equilibria are locally efficient
(this covers the normal form case: $f(x) = \frac{1}{2}x'Ax$)

4.2 Evolutionarily stable states and contractive games

4.2.1 Evolutionarily stable states (Maynard Smith and Price (1973))

The original definition of ESS (stated for population games):

(1a) x is a Nash equilibrium: $(y - x)'F(x) \leq 0$ for all $y \in X$.

(1b) There is a neighborhood O of x such that for all $y \in O \setminus \{x\}$,

$$(y - x)'F(x) = 0 \text{ implies that } (y - x)'F(y) < 0.$$

An equivalent definition: an ESS is an (infinitesimal) local invader:

(2) There is a neighborhood O of x such that $(y - x)'F(y) < 0$ for all $y \in O \setminus \{x\}$.

The classic local stability result:

Theorem (Taylor and Jonker (1978), HSS (1979), Zeeman (1980))**.**

An ESS is locally stable under the replicator dynamic.

Questions:

Q1. Can we establish local stability of ESS under other dynamics?

Q2. Does the definition of ESS suggest a class of games
with good global stability properties?

For Q2: We derive a local property of payoffs near an interior ESS x,
and then impose this condition globally.

interior Nash equilibrium:	$(y - x)'F(x) = 0$
(infinitesimal) global invasion:	$(y - x)'F(y) < 0$

Questions:

Q1. Can we establish local stability of ESS under other dynamics?

Q2. Does the definition of ESS suggest a class of games with good global stability properties?

For Q2: We derive a local property of payoffs near an interior ESS x, and then impose this condition globally.

		$z \equiv y - x$
interior Nash equilibrium:	$(y-x)'F(x) = 0$	$z'F(x) = 0$
(infinitesimal) global invasion:	$(y-x)'F(y) < 0$	$z'F(x+z) < 0$

Questions:

Q1. Can we establish local stability of ESS under other dynamics?

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For Q2: We derive a local property of payoffs near an interior ESS x,
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		$z \equiv y - x$
interior Nash equilibrium:	$(y-x)'F(x) = 0$	$z'F(x) = 0$
(infinitesimal) global invasion:	$(y-x)'F(y) < 0$	$z'F(x+z) < 0$
subtract:		$z'(F(x+z) - F(x)) < 0$

Questions:

Q1. Can we establish local stability of ESS under other dynamics?

Q2. Does the definition of ESS suggest a class of games
with good global stability properties?

For Q2: We derive a local property of payoffs near an interior ESS x,
and then impose this condition globally.

		$z \equiv y - x$
interior Nash equilibrium:	$(y - x)'F(x) = 0$	$z'F(x) = 0$
(infinitesimal) global invasion:	$(y - x)'F(y) < 0$	$z'F(x + z) < 0$
subtract:		$z'(F(x + z) - F(x)) < 0$
take $\lvert z\rvert \to 0$:		$z'DF(x)z < 0$

4.2.2 Contractive games: definition and characterization

The population game $F\colon X \to \mathbb{R}^n$ is a (strictly) contractive game if

(C) $\quad (y-x)'(F(y)-F(x)) < 0$ for all $x, y \in X$.

(a.k.a.: stable game, negative definite game, monotone vector field.)

Let $TX = \{z \in \mathbb{R}^n \colon \sum_i z_i = 0\}$ be the set of vectors tangent to X.

If F is smooth, it is strictly contractive if and only
if it satisfies self-defeating externalities:

(SD) $\quad z'DF(x)z < 0$ for all $z \in TX$ and $x \in X$.

When $z = e_j - e_i$, (SD) becomes $\dfrac{\partial F_j}{\partial(e_j - e_i)}(x) \leq \dfrac{\partial F_i}{\partial(e_j - e_i)}(x)$.

(C) $(y - x)'(F(y) - F(x)) \leq 0$ for all $x, y \in X$.

(SD) $z'DF(x)z < 0$ for all $z \in TX$ and $x \in X$.

Intuition 1: If F is a potential game, (C) and (SD) say that f is strictly concave.

(C) $\quad (y - x)'(F(y) - F(x)) \leq 0$ for all $x, y \in X$.

(SD) $\quad z'DF(x)z < 0$ for all $z \in TX$ and $x \in X$.

Intuition 1: If F is a potential game, (C) and (SD) say that f is strictly concave.

Intuition 2: From states in $\text{int}(X)$, consider "following the payoff vectors":

$\dot{x} = \Phi F(x)$ (where Φ projects $\mathbb{R}^n$ orthogonally onto TX).

Run from two initial states x_0 and y_0:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|y_t - x_t\right|^2 = 2(y_t - x_t)'(\dot{y}_t - \dot{x}_t) = 2(y_t - x_t)'(F(y_t) - F(x_t)) \leq 0.$$

Following the payoff vectors brings states closer together.

4.2.3 Examples

ex.: Rock-Paper-Scissors

$$A = \begin{pmatrix} 0 & -\ell & w \\ w & 0 & -\ell \\ -\ell & w & 0 \end{pmatrix}.$$

Here $w > 0$ and $\ell > 0$ represent the benefit from a win and the cost of a loss.

$w = \ell$ is (standard) RPS, $w > \ell$ is good RPS, $w < \ell$ is bad RPS.

In all cases, the unique Nash equilibrium is $x^* = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

When is $F(x) = Ax$ contractive?

ex. Wars of attrition

ex. (Perturbed) concave potential games

4.2.4 Equilibrium in contractive games

We call x a globally evolutionarily stable state of F ($x \in GESS(F)$) if

$$(y - x)' F(y) < 0 \text{ for all } y \in X \smallsetminus \{x\}.$$

$x \in NE(F)$: $\quad (x - y)'F(x) \geq 0$ for all $y \in X$

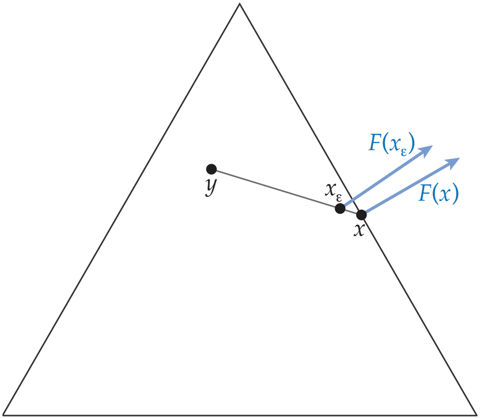
$x \in GESS(F)$: $\quad (x - y)'F(y) > 0$ for all $y \neq x$

Proposition. *If $x \in GESS(F)$ (so a GESS exists), then $NE(F) = \{x\}$.*

$x \in NE(F)$: $\quad (x - y)'F(x) \geq 0$ for all $y \in X$

$x \in GESS(F)$: $\quad (x - y)'F(y) > 0$ for all $y \neq x$

Proposition. *If* $x \in GESS(F)$ *(so a GESS exists), then* $NE(F) = \{x\}$.



Proof. Fix x, $y \neq x$.
Let $x_\varepsilon = \varepsilon y + (1 - \varepsilon)x$.
If $x \in GESS(F)$, then

$$\begin{aligned} & (x - x_\varepsilon)'F(x_\varepsilon) > 0 \\ \Rightarrow\ & (x - y)'F(x_\varepsilon) > 0 \\ \Rightarrow\ & (x - y)'F(y) \geq 0 \end{aligned}$$

So $x \in NE(F)$.

And $x \in GESS(F) \Rightarrow$
no $y \neq x$ is a BR to itself.

Theorem.

If F is a strictly contractive game, then $GNSS(F)$ is a singleton and coincides with $NE(F)$.

Proof:

strictly contractive game: $(y - x)'(F(y) - F(x)) < 0$ for all $y \in X$

Nash equilibrium: $(y - x)'F(x) < 0$ for all $y \in X$

add to get GESS: $(y - x)'F(y) < 0$ for all $y \in X$

One can prove existence of Nash equilibrium in contractive games using the minmax theorem and a compactness argument.

4.2.5 Global convergence in contractive games

dynamic	global Lyapunov function for contractive games
replicator	$H_{x^*}(x) = \sum_{i \in \operatorname{supp}(x^*)} x_i^* \log \frac{x_i^*}{x_i}$
Smith	$\Psi(x) = \frac{1}{2} \sum_{i \in S} \sum_{j \in S} x_i [F_j(x) - F_i(x)]_+^2$
best response	$G(x) = \max_{i \in S} (F_i(x) - \bar{F}(x))$
logit	$\tilde{G}(x) = \max_{y \in \operatorname{int}(X)} \left(y' \hat{F}(x) - \eta \sum_{i \in S} y_i \log y_i \right) + \eta \sum_{i \in S} x_i \log x_i$

Proof for replicator only uses the fact that x^* is a GESS.

(See HSS (1979), Zeeman (1980).)

Global convergence in contractive games

dynamic	global Lyapunov function for contractive games
replicator	$H_{x^*}(x) = \sum_{i \in \mathrm{supp}(x^*)} x_i^* \log \frac{x_i^*}{x_i}$
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Proofs for other dynamics combine (SD) with (PC).

ex.: for the best response dynamic,

$$\dot{G}(x_t) = \dot{x}_t' \, DF(x_t) \, \dot{x}_t - F(x_t)' \dot{x}_t.$$

(See Hofbauer (2000), Hofbauer-Sandholm (2007, 2009), Zusai (2018))

4.2.6 Local stability of ESS

Imitative dynamics

Two approaches to proving local stability of ESS for general imitative dynamics:

1. Linearization (Cressman (1997))
2. Construction of Lyapunov functions defined by “distances” from the ESS (Mertikopoulos-Sandholm (2018))

Direct dynamics

If a game has an interior ESS, the game is “locally contractive” near the ESS.

Thus the global Lyapunov functions for contractive games
are local Lyapunov functions for interior ESS.

By adding appropriate penalty terms,
we obtain local Lyapunov functions for boundary ESS.

(See Sandholm (2010), Zusai (2018).)

5. Nonconvergence

Potential games and contractive games permit global convergence results.

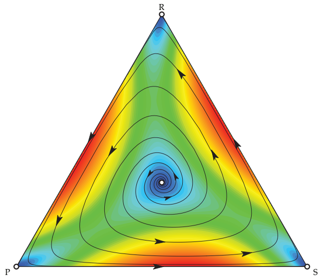
In general, there is no reason to expect global convergence;
disequilibrium play may persist indefinitely.

Evolutionary dynamics cannot provide
an unconditional justification of equilibrium play.

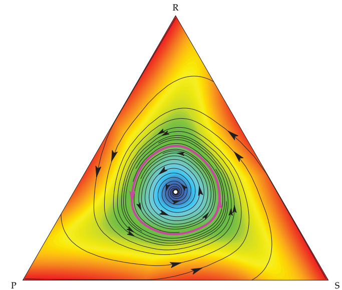
5.1 Examples

Example (Bad Rock-Paper-Scissors).

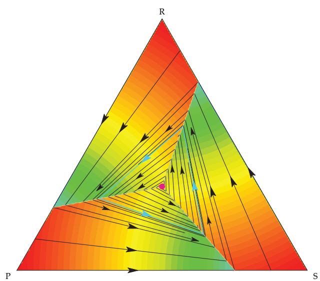
$$A = \begin{pmatrix} 0 & -\ell & w \\ w & 0 & -\ell \\ -\ell & w & 0 \end{pmatrix}, \quad \text{where } \ell > w > 0.$$



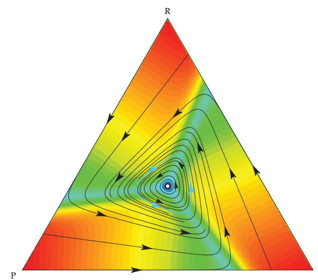
(i) replicator



(ii) Smith



(iii) best response



(iv) logit(.1)

Five basic deterministic dynamics in bad Rock-Paper-Scissors ($l = 2$, $w = 1$).

Example (The Hofbauer-Swinkels game).

$$A^\varepsilon = \begin{pmatrix} 0 & 0 & -1 & \varepsilon \\ \varepsilon & 0 & 0 & -1 \\ -1 & \varepsilon & 0 & 0 \\ 0 & -1 & \varepsilon & 0 \end{pmatrix}$$

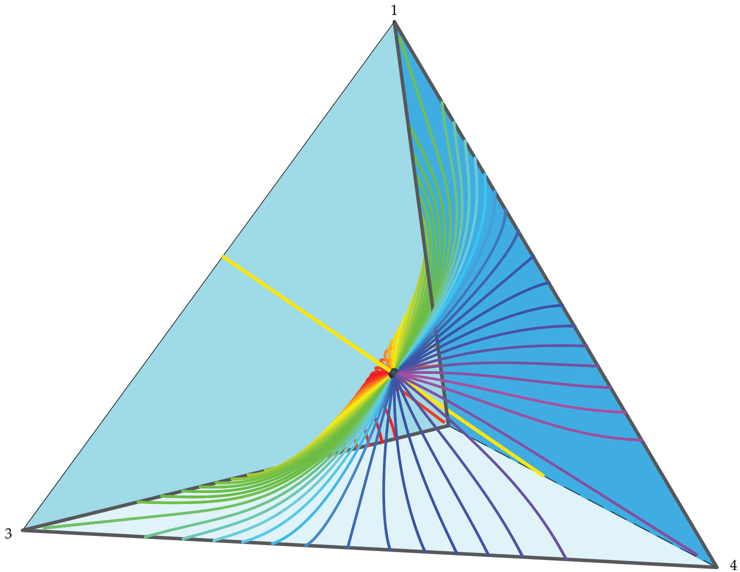
When $\varepsilon = 0$, the payoff matrix $A^\varepsilon = A^0$ is symmetric.

$F^0(x) = A^0 x$ is a potential game with potential function $f(x) = \frac{1}{2}x' A^0 x = -x_1 x_3 - x_2 x_4$.

f is minimized at $v = (\frac{1}{2}, 0, \frac{1}{2}, 0)$ and $w = (0, \frac{1}{2}, 0, \frac{1}{2})$;

f has a saddle point at Nash equilibrium $x^* = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$;

f is maximized on boundary polygon γ ($e_1 \to e_2 \to e_3 \to e_4 \to e_1$).



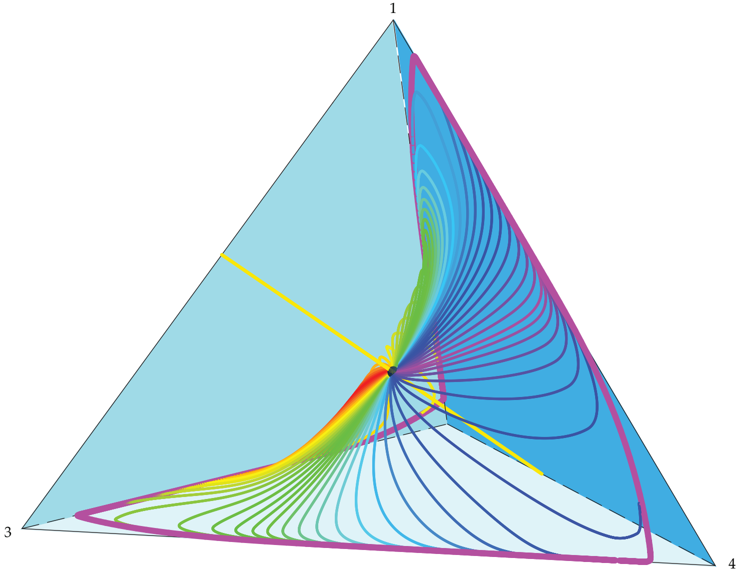
The Smith dynamic in A^0.

$$A^\varepsilon = \begin{pmatrix} 0 & 0 & -1 & \varepsilon \\ \varepsilon & 0 & 0 & -1 \\ -1 & \varepsilon & 0 & 0 \\ 0 & -1 & \varepsilon & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}.$$

The attractor γ of V^0 continues to an attractor γ^ε of V^ε.

This attractor attracts solutions to V^ε from many initial conditions.

But the unique Nash equilibrium of A^ε is x^*!



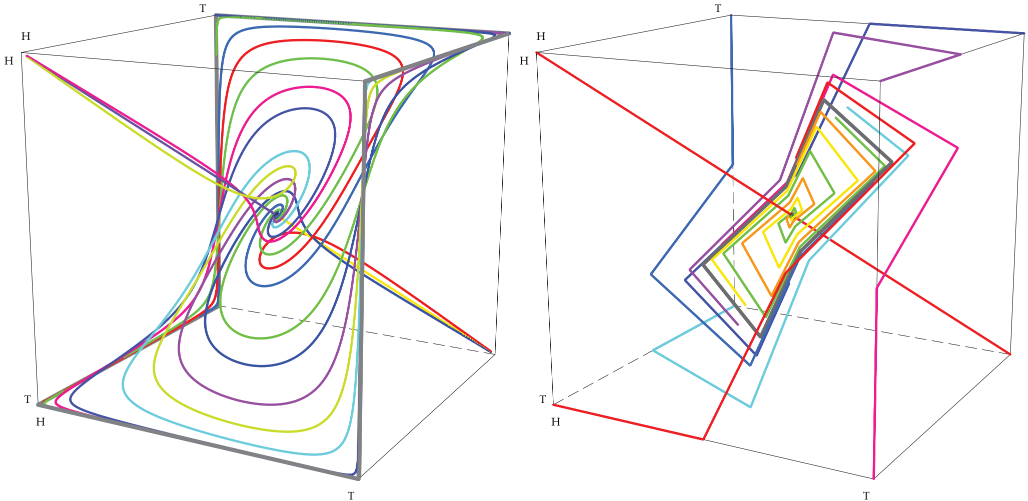
The Smith dynamic in A^{ε}, $\varepsilon = \frac{1}{10}$.

Example. Mismatching Pennies is a three-player normal form game.

Each player has two strategies, Heads and Tails.

Player p receives a payoff of 1 for choosing a different strategy than player $p + 1$ and a payoff of 0 otherwise (where players are indexed modulo 3).

The unique Nash equilibrium of this game has each player play each of his strategies with equal probability.



replicator

best response

Proposition (Hart & Mas Colell (2003))**.**

Let V be a smooth evolutionary dynamic that satisfies Nash stationarity (NS)*.*

Let F be Mismatching Pennies, and suppose that the unique Nash equilibrium x^* of F is a hyperbolic rest point of $\dot{x} = V^F(x)$.

Then x^* is unstable under V^F, and there is an open, dense, full measure set of initial conditions from which solutions to V^F do not converge.

Example (Chaotic dynamics).

In population games with four or more strategies,
solutions can converge to chaotic attractors.

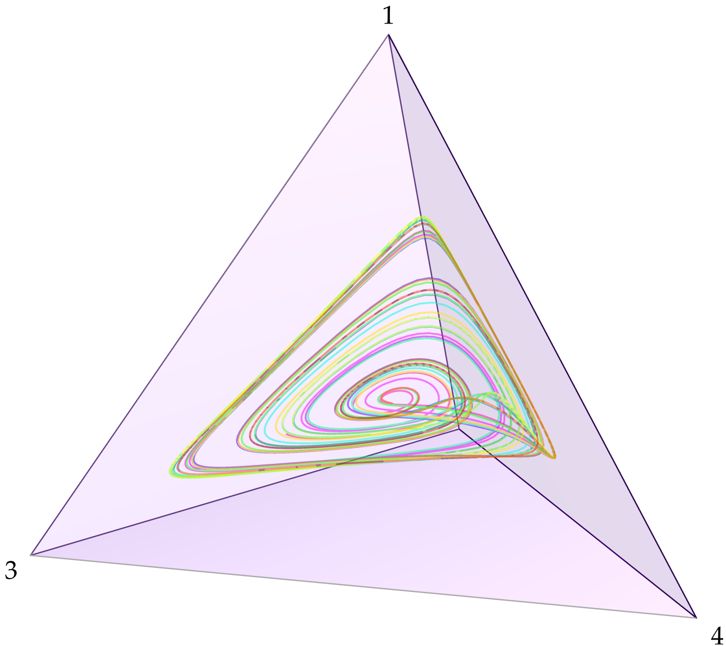
These exhibit sensitive dependence on initial conditions:
solution trajectories starting close together
move apart at an exponential rate.

Chaotic attractors are distinguished by their intricate appearance.

ex. (ACT (1980), Skyrms (1992))

Consider the replicator dynamic in

$$A = \begin{pmatrix} 0 & -12 & 0 & 22 \\ 20 & 0 & 0 & -10 \\ -21 & -4 & 0 & 35 \\ 10 & -2 & 2 & 0 \end{pmatrix}$$


1
3
4

5.2 Survival of strictly dominated strategies

(Berger-Hofbauer (2006), Hofbauer-Sandholm (2011))

Strictly dominated strategies are eliminated
under imitative dynamics and the best response dynamic.

How robust are these results?

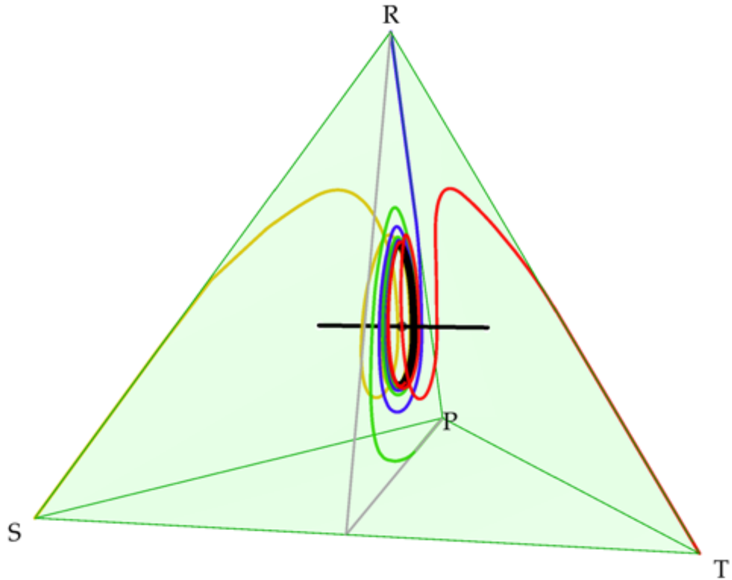
Consider the Smith dynamic for “bad RPS with a twin”:

$$A = \begin{pmatrix} 0 & -2 & 1 & 1 \\ 1 & 0 & -2 & -2 \\ -2 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 \end{pmatrix}$$

The set of Nash equilibria, $NE(A) = \{x^* \in X \colon x^* = (\frac{1}{3}, \frac{1}{3}, c, \frac{1}{3} - c)\}$, is a repellor.

Away from $NE(A)$, strategies lose agents at rates proportional to usage levels, but gain players at rates that depend on payoffs.

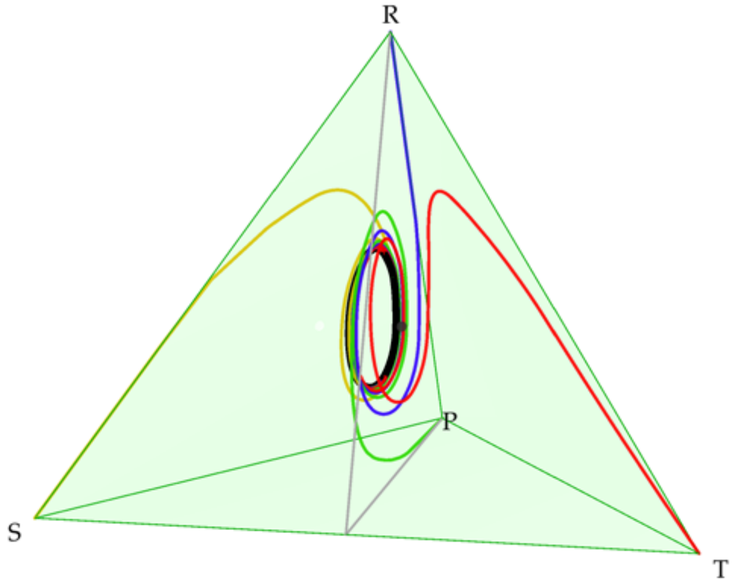
Thus the proportions of players choosing the twin strategies are equalized, converging to plane $\{x \in X : x_S = x_T\}$.



The Smith dynamic in bad RPS with a twin.

Now consider the Smith dynamic in “bad RPS with a feeble twin”,

$$F^d(x) = A^d x = \begin{pmatrix} 0 & -2 & 1 & 1 \\ 1 & 0 & -2 & -2 \\ -2 & 1 & 0 & 0 \\ -2-d & 1-d & -d & -d \end{pmatrix} \begin{pmatrix} x_R \\ x_P \\ x_S \\ x_T \end{pmatrix},$$



The Smith dynamic in bad RPS with a feeble twin, $d = \frac{1}{10}$.

A similar construction works for most traditional dynamics, other than imitative dynamics and the best response dynamic.

How can strictly dominated strategies survive?

1. Agents base their decisions on the strategies' present payoffs, and not on knowledge about payoff functions.

2. Nonconverge is crucial.

6. Sampling dynamics for normal form games

The canonical dynamics and their families are based on revision protocols that use exact information about payoffs.

This leads to properties like (PC) and (NS),
and to global convergence results in nice games.

It is sometimes more realistic to assume that agents only have information from samples, or from limited testing of strategies.

The resulting dynamics can behave quite differently than the canonical ones.

6.1 Sampling best response dynamics and equilibrium selection

(Sandholm (2001), KDV (2002), OST (2015))

A revising agent obtains information by drawing
a sample of size k from the population.

The agent then plays a best response
to the empirical distribution of strategies in his sample.

Observation.

If strategy i is a strict equilibrium of F, then pure state e_i is a rest point of any sampling best response dynamic V.

In game A, strategy i is p-dominant (Kajii-Morris (1997))
if i is the unique best response at any state x with $x_i \geq \frac{1}{k}$.

Decreasing p makes p-dominance more demanding:
1-dominance = strict equilibrium;
0-dominance = strictly dominant strategy.

Theorem.

Suppose $A \in \mathbb{R}^{2\times 2}$, and that strategy i is $\frac{1}{k}$-dominant, where $k \geq 2$.
Under the k-sampling best response dynamic,
if play begins at a state with $x_i > 0$
it converges to the pure state $x_i = 1$.

Theorem.

Suppose $A \in \mathbb{R}^{2\times 2}$, and that strategy i is $\frac{1}{k}$-dominant, where $k \geq 2$.

Under the k-sampling best response dynamic,
if play begins at a state with $x_i > 0$
it converges to the pure state e_i.

Proof. Suppose i is $\frac{1}{k}$-dominant (but not strictly dominant).

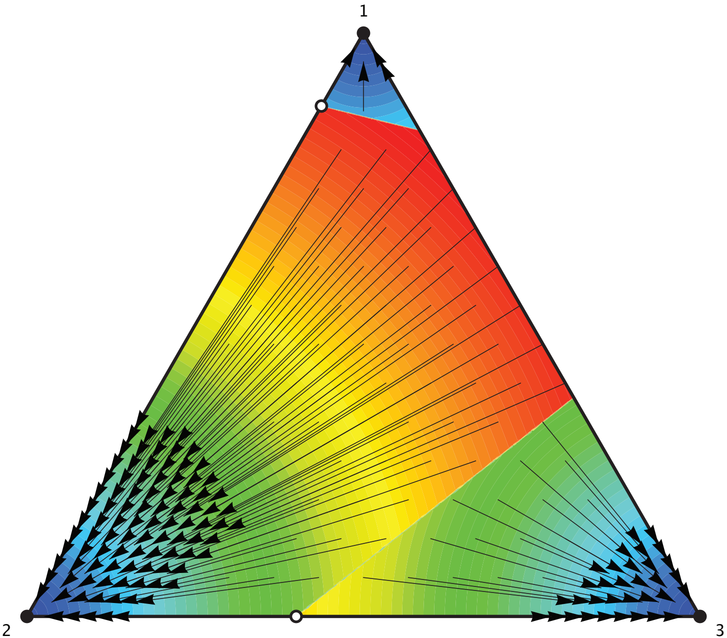
Consider changes in the use of strategy $j \neq i$:

Gross outflow from j (due to revision opportunities): x_j.

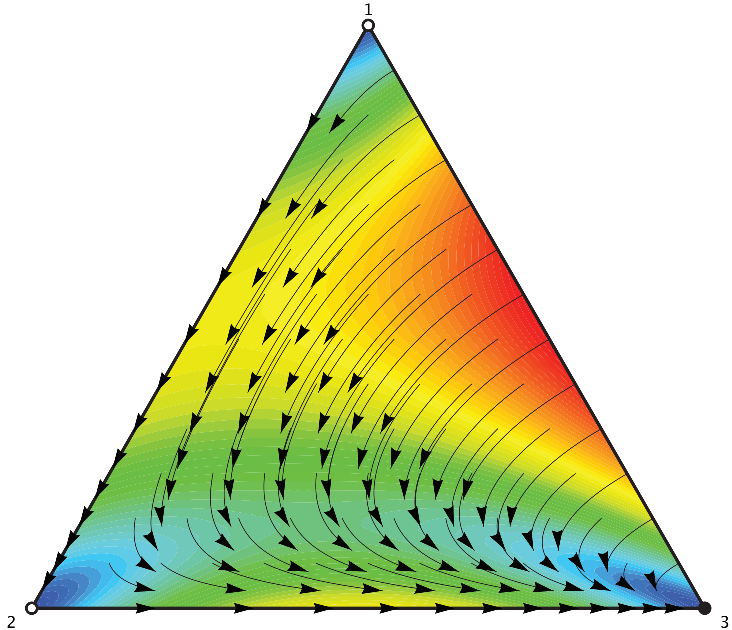
Gross inflow into j (because j is best response to sample): $(x_j)^k$.

$\therefore \quad \dot{x}_j = (x_j)^k - x_j \leq 0.$

This logic extends to iterated versions of p-dominance.



The best response dynamic in Young's game, $A = \begin{pmatrix} 6 & 0 & 0 \\ 5 & 7 & 5 \\ 0 & 5 & 8 \end{pmatrix}$.



The 2-sampling best response dynamic in Young's game, $A = \begin{pmatrix} 6 & 0 & 0 \\ 5 & 7 & 5 \\ 0 & 5 & 8 \end{pmatrix}$.

6.2 Best experienced payoff dynamics

(Sethi (2000), SII (2018))

Under SBR dynamics, agents take a sample of opponents' strategies
and play a best response to the empirical distribution of strategies in the sample.

This requires players to know the payoff matrix
and engage in counterfactual reasoning.

One can define a less demanding model based only on experienced payoffs:

Agents test strategies by playing them against randomly chosen opponents,
and then switch to the strategy that performed best during testing.

Some basic properties of BEP dynamics

1. Strict equilibria are rest points

2. Strictly dominated strategies need not be eliminated.

$$A = \begin{pmatrix} 2 & 5 & 8 \\ 1 & 4 & 7 \\ 0 & 3 & 6 \end{pmatrix}$$

Tomorrow:
In the Centipede game, BEP dynamics lead to stable cooperative play.

6.3 Research idea

Sampling dynamics respect strict equilibria.

One can generalize strict equilibria using set-valued solution concepts, namely curb sets and variations (Basu-Weibull (1991), Voorneveld (2004)).

Curb-like sets are known to have good stability properties under certain imitative dynamics (Ritzberger-Weibull (1995)) and for versions of best response dynamics (Hurkens (1995), BHK (2013)).

What properties be established for SBR dynamics and BEP dynamics?

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